Lecture Notes in Mathematics 1848

Editors:
J.–M. Morel, Cachan
F. Takens, Groningen
B. Teissier, Paris

Subseries:
Fondazione C.I.M.E., Firenze
Adviser: Pietro Zecca

M. Abate J. E. Fornaess X. Huang
J.-P. Rosay A. Tumanov

Real Methods in Complex and CR Geometry

Lectures given at the
C.I.M.E. Summer School
held in Martina Franca, Italy,
June 30 – July 6, 2002

Editors: D. Zaitsev
G. Zampieri

Fondazione
C.I.M.E.

Springer

Editors and Authors

Marco Abate
Department of Mathematics
University of Pisa
via Buonarroti 2
56127 Pisa Italy

e-mail: abate@dm.unipi.it

John Erik Fornaess
Department of Mathematics
University of Michigan
East Hall, Ann Arbor
MI 48109, U.S.A.

e-mail: fornaess@umich.edu

Xiaojun Huang
Department of Mathematics
Rutgers University
New Brunswick
N.J. 08903, U.S.A.

e-mail: huangx@math.rutgers.edu

Jean-Pierre Rosay
Department of Mathematics
University of Wisconsin
Madison, WI 53706-1388, USA

e-mail: jrosay@math.wisc.edu

Alexander Tumanov
Department of Mathematics
University of Illinois
1409 W. Green Street
Urbana, IL 61801, U.S.A.

e-mail: tumanov@math.uiuc.edu

Dmitri Zaitsev
School of Mathematics
Trinity College
University of Dublin
Dublin 2, Ireland

e-mail: zaitsev@maths.tcd.ie

Giuseppe Zampieri
Department of Mathematics
University of Padova
via Belzoni 7
35131 Padova, Italy

e-mail: zampieri@math.unipd.it

Library of Congress Control Number: 2004094684

Mathematics Subject Classification (2000): 32V05, 32V40, 32A40, 32H50 32VB25, 32V35

ISSN 0075-8434
ISBN 3-540-22358-4 Springer Berlin Heidelberg New York
DOI: 10.1007/b98482

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

Springer is a part of Springer Science+Business Media
springeronline.com
© Springer-Verlag Berlin Heidelberg 2004
Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: Camera-ready TEX output by the authors

Printed on acid-free paper
41/3142/du - 5 4 3 2 1 0

Preface

The C.I.M.E. Session "Real Methods in Complex and CR Geometry" was held in Martina Franca (Taranto), Italy, from June 30 to July 6, 2002. Lecture series were given by:

M. Abate: *Angular derivatives in several complex variables*
J. E. Fornaess: *Real methods in complex dynamics*
X. Huang: *On the Chern-Moser theory and rigidity problem for holomorphic maps*
J. P. Rosay: *Theory of analytic functionals and boundary values in the sense of hyperfunctions*
A. Tumanov: *Extremal analytic discs and the geometry of CR manifolds*

These proceedings contain the expanded versions of these five courses. In their lectures the authors present at a level accessible to graduate students the current state of the art in classical fields of the geometry of complex manifolds (Complex Geometry) and their real submanifolds (CR Geometry). One of the central questions relating both Complex and CR Geometry is the behavior of holomorphic functions in complex domains and holomorphic mappings between different complex domains at their boundaries. The existence problem for boundary limits of holomorphic functions (called boundary values) is addressed in the Julia-Wolff-Caratheodory theorem and the Lindelöf principle presented in the lectures of M. Abate. A very general theory of boundary values of (not necessarily holomorphic) functions is presented in the lectures of J.-P. Rosay. The boundary values of a holomorphic function always satisfy the tangential Cauchy-Riemann (CR) equations obtained by restricting the classical CR equations from the ambient complex manifold to a real submanifold. Conversely, given a function on the boundary satisfying the tangential CR equations (a CR function), it can often be extended to a holomorphic function in a suitable domain. Extension problems for CR mappings are addressed in the lectures of A. Tumanov via the powerful method of the extremal and stationary discs. Another powerful method coming from the formal theory and inspired by the work of Chern and Moser is presented in the lectures of X. Huang addressing the existence questions for CR maps. Finally, the dynamics of holomorphic maps in several complex variables is the topic of the lectures of J. E. Fornaess linking Complex Geometry and its methods with the theory of Dynamical Systems.

We hope that these lecture notes will be useful not only to experienced readers but also to the beginners aiming to learn basic ideas and methods in these fields.

We are thankful to the authors for their beautiful lectures, all participants from Italy and abroad for their attendance and contribution and last but not least CIME for providing a charming and stimulating atmosphere during the school.

Dmitri Zaitsev and Giuseppe Zampieri

CIME's activity is supported by:

Ministero degli Affari Esteri - Direzione Generale per la Promozione e la Cooperazione - Ufficio V;
Consiglio Nazionale delle Ricerche;
E.U. under the Training and Mobility of Researchers Programme.

Contents

Angular Derivatives in Several Complex Variables

Marco Abate

Dipartimento di Matematica, Università di Pisa
Via Buonarroti 2, 56127 Pisa, Italy
abate@dm.unipi.it

1 Introduction

A well-known classical result in the theory of one complex variable, due to Fatou [Fa], says that a bounded holomorphic function f defined in the unit disk Δ admits non-tangential limit at almost every point $\sigma \in \partial\Delta$. As satisfying as it is from several points of view, this theorem leaves open the question of whether the function f admits non-tangential limit at a *specific* point $\sigma_0 \in \partial\Delta$.

Of course, one needs to make some assumptions on the behavior of f near the point σ_0; the aim is to find the weakest possible assumptions. In 1920, Julia [Ju1] identified the right hypothesis: assuming, without loss of generality, that the image of the bounded holomorphic function is contained in the unit disk then Julia's assumption is

$$\liminf_{\zeta \to \sigma_0} \frac{1 - |f(\zeta)|}{1 - |\zeta|} < +\infty. \tag{1}$$

In other words, $f(\zeta)$ must go to the boundary as fast as ζ (as we shall show, it cannot go to the boundary any faster, but it might go slower). Then Julia proved the following

Theorem 1.1 (Julia) *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be a bounded holomorphic function, and take $\sigma \in \partial\Delta$ such that*

$$\liminf_{\zeta \to \sigma} \frac{1 - |f(\zeta)|}{1 - |\zeta|} = \beta < +\infty$$

for some $\beta \in \mathbb{R}$. Then $\beta > 0$ and f has non-tangential limit $\tau \in \partial\Delta$ at σ.

As we shall see, the proof is just a (clever) application of Schwarz-Pick lemma. The real breakthrough in this theory is due to Wolff [Wo] in 1926 and Carathéodory [C1] in 1929: if f satisfies 1 at σ then the derivative f' too admits finite non-tangential limit at σ — and this limit can be computed explicitely. More precisely:

M. Abate et al.: LNM 1848, D. Zaitsev and G. Zampieri (Eds.), pp. 1–47, 2004.
© Springer-Verlag Berlin Heidelberg 2004

Theorem 1.2 (Wolff-Carathéodory) *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be a bounded holomorphic function, and take $\sigma \in \partial\Delta$ such that*

$$\liminf_{\zeta \to \sigma} \frac{1 - |f(\zeta)|}{1 - |\zeta|} = \beta < +\infty$$

for some $\beta > 0$. Then both the incremental ratio

$$\frac{f(\zeta) - \tau}{\zeta - \sigma}$$

and the derivative f' have non-tangential limit $\beta\tau\bar{\sigma}$ at σ, where $\tau \in \partial\Delta$ is the non-tangential limit of f at σ.

Theorems 1.1 and 1.2 are collectively known as the *Julia - Wolff - Cara–théodory theorem.* The aim of this survey is to present a possible way to generalize this theorem to bounded holomorphic functions of several complex variables.

The main point to be kept in mind here is that, as first noticed by Korányi and Stein (see, e.g., [St]) and later theorized by Krantz [Kr1], the right kind of limit to consider in studying the boundary behavior of holomorphic functions of several complex variables depends on the geometry of the domain, and it is usually stronger than the non-tangential limit. To better stress this interdependence between analysis and geometry we decided to organize this survey as a sort of template that the reader may apply to the specific cases s/he is interested in.

More precisely, we shall single out a number of geometrical hypotheses (usually expressed in terms of the Kobayashi intrinsic distance of the domain) that when satisfied will imply a Julia-Wolff-Carathéodory theorem. This approach has the advantage to reveal the main ideas in the proofs, unhindered by the technical details needed to verify the hypotheses. In other words, the hard computations are swept under the carpet (i.e., buried in the references), leaving the interesting patterns *over* the carpet free to be examined.

Of course, the hypotheses can be satisfied: for instance, all of them hold for strongly pseudoconvex domains, convex domains with C^ω boundary, convex circular domains of finite type, and in the polydisk; but most of them hold in more general domains too. And one fringe benefit of the approach chosen for this survey is that as soon as somebody proves that the hypotheses hold for a specific domain, s/he gets a Julia-Wolff-Carathéodory theorem in that domain for free. Indeed, this approach has already uncovered new results: to the best of my knowledge, Theorem 4.2 in full generality and Proposition 4.8 have not been proved before.

So in Section 1 of this survey we shall present a proof of the Julia-Wolff-Carathéodory theorem suitable to be generalized to several complex variables. It will consist of three steps:

(a) A proof of Theorem 1.1 starting from the Schwarz-Pick lemma.
(b) A discussion of the Lindelöf principle, which says that if a (K-)bounded holomorphic function has limit restricted to a curve ending at a boundary point then it has the same limit restricted to any non-tangential curve ending at that boundary point.
(c) A proof of the Julia-Wolff-Carathéodory theorem obtained by showing that the incremental ratio and the derivative satisfy the hypotheses of the Lindelöf principle.

Then the next three sections will describe a way of performing the same three steps in a several variables context, providing the template mentioned above.

Finally, a few words on the literature. As mentioned before, Theorem 1.1 first appeared in [Ju1], and Theorem 1.2 in [Wo]. The proof we shall present here is essentially due to Rudin [Ru, Section 8.5]; other proofs and one-variable generalizations can be found in [A3], [Ah], [C1, 2], [J], [Kom], [LV], [Me], [N], [Po], [T] and references therein.

As far as I know, the first several variables generalizations of Theorem 1.1 were proved by Minialoff [Mi] for the unit ball $B^2 \subset \mathbb{C}^2$, and then by Hervé [He] in B^n. The general form we shall discuss originates in [A2]. For some other (finite and infinite dimensional) approaches see [Ba], [M], [W], [R], [Wł1] and references therein.

The one-variable Lindelöf principle has been proved by Lindelöf [Li1, 2]; see also [A3, Theorem 1.3.23], [Ru, Theorem 8.4.1], [Bu, 5.16, 5.56, 12.30, 12.31] and references therein. The first important several variables version of it is due to Čirka [Č]; his approach has been further pursued in [D1, 2], [DZ] and [K]. A different generalization is due to Cima and Krantz [CK] (see also [H1, 2]), and both inspired the presentation we shall give in Section 3 (whose ideas stem from [A2]).

A first tentative extension of the Julia-Wolff-Carathéodory theorem to bounded domains in $\mathbb{C}^2$ is due to Wachs [W]. Hervé [He] proved a preliminary Julia-Wolff-Carathéodory theorem for the unit ball of $\mathbb{C}^n$ using non-tangential limits and considering only incremental ratioes; the full statement for the unit ball is due to Rudin [Ru, Section 8.5]. The Julia-Wolff-Carathéodory theorem for strongly convex domains is in [A2]; for strongly pseudoconvex domains in [A4]; for the polydisk in [A5] (see also Jafari [Ja], even though his statement is not completely correct); for convex domains of finite type in [AT2]. Furthermore, Julia-Wolff-Carathéodory theorems in infinite-dimensional Banach and Hilbert spaces are discussed in [EHRS], [F], [MM], [SW], [Wł2, 3, 4], [Z] and references therein.

Finally, I would also like to mention the shorter survey [AT1], written, as well as the much more substantial paper [AT2], with the unvaluable help of Roberto Tauraso.

2 One Complex Variable

We already mentioned that Theorem 1.1 is a consequence of the classical Schwarz-Pick lemma. For the sake of completeness, let us recall here the relevant definitions and statements.

Definition 2.1 *The Poincaré metric on Δ is the complete Hermitian metric κ_Δ^2 of constant Gaussian curvature -4 given by*

$$\kappa_\Delta^2(\zeta) = \frac{1}{(1-|\zeta|^2)^2}\, dz\, d\bar{z}.$$

The Poincaré distance ω on Δ is the integrated distance associated to κ_Δ.

It is easy to prove that

$$\omega(\zeta_1,\zeta_2) = \tfrac{1}{2}\log\frac{1+\left|\frac{\zeta_1-\zeta_2}{1-\bar{\zeta}_2\zeta_1}\right|}{1-\left|\frac{\zeta_1-\zeta_2}{1-\bar{\zeta}_2\zeta_1}\right|}.$$

For us the main property of the Poincaré distance is the classical Schwarz-Pick lemma:

Theorem 2.2 (Schwarz-Pick) *The Poincaré metric and distance are contracted by holomorphic self-maps of the unit disk. In other words, if $f \in \mathrm{Hol}(\Delta,\Delta)$ then*

$$\forall \zeta \in \Delta \qquad f^*(\kappa_\Delta^2)(\zeta) \le \kappa_\Delta^2(\zeta) \tag{2}$$

and

$$\forall \zeta_1,\zeta_2 \in \Delta \qquad \omega\big(f(\zeta_1), f(\zeta_2)\big) \le \omega(\zeta_1,\zeta_2). \tag{3}$$

Furthermore, equality in 2 for some $\zeta \in \Delta$ or in 3 for some $\zeta_1 \ne \zeta_2$ occurs iff f is a holomorphic automorphism of Δ.

A first easy application of this result is the fact that the lim inf in 1 is always positive (or $+\infty$). But let us first give it a name.

Definition 2.3 *Let $f \in \mathrm{Hol}(\Delta,\Delta)$ be a holomorphic self-map of Δ, and $\sigma \in \partial\Delta$. Then the* boundary dilation coefficient *$\beta_f(\sigma)$ of f at σ is given by*

$$\beta_f(\sigma) = \liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|}.$$

If it is finite and equal to $\beta > 0$ we shall say that f is β-Julia at σ.

Then

Corollary 2.4 *For any* $f \in \mathrm{Hol}(\Delta, \Delta)$ *we have*

$$\frac{1-|f(\zeta)|}{1-|\zeta|} \geq \frac{1-|f(0)|}{1+|f(0)|} > 0 \tag{4}$$

for all $\zeta \in \Delta$*; in particular,*

$$\beta_f(\sigma) \geq \frac{1-|f(0)|}{1+|f(0)|} > 0$$

for all $\sigma \in \partial\Delta$*.*

Proof. The Schwarz-Pick lemma yields

$$\omega\big(0, f(\zeta)\big) \leq \omega\big(0, f(0)\big) + \omega\big(f(0), f(\zeta)\big) \leq \omega\big(0, f(0)\big) + \omega(0, \zeta),$$

that is

$$\frac{1+|f(\zeta)|}{1-|f(\zeta)|} \leq \frac{1+|f(0)|}{1-|f(0)|} \cdot \frac{1+|\zeta|}{1-|\zeta|} \tag{5}$$

for all $\zeta \in \Delta$. Let $a = (|f(0)| + |\zeta|)/(1 + |f(0)||\zeta|)$; then the right-hand side of 5 is equal to $(1+a)/(1-a)$. Hence $|f(\zeta)| \leq a$, that is

$$1 - |f(\zeta)| \geq (1-|\zeta|)\frac{1-|f(0)|}{1+|f(0)||\zeta|} \geq (1-|\zeta|)\frac{1-|f(0)|}{1+|f(0)|}$$

for all $\zeta \in \Delta$, as claimed. □

The main step in the proof of Theorem 1.1 is known as *Julia's lemma*, and it is again a consequence of the Schwarz-Pick lemma:

Theorem 2.5 (Julia) *Let* $f \in \mathrm{Hol}(\Delta, \Delta)$ *and* $\sigma \in \partial\Delta$ *be such that*

$$\liminf_{\zeta \to \sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \beta < +\infty.$$

Then there exists a unique $\tau \in \partial\Delta$ *such that*

$$\frac{|\tau - f(\zeta)|^2}{1-|f(\zeta)|^2} \leq \beta \frac{|\sigma - \zeta|^2}{1-|\zeta|^2}. \tag{6}$$

Proof. The Schwarz-Pick lemma yields

$$\left| \frac{f(\zeta) - f(\eta)}{1 - \overline{f(\eta)} f(\zeta)} \right| \leq \left| \frac{\zeta - \eta}{1 - \bar{\eta}\zeta} \right|$$

and thus

$$\frac{|1 - \overline{f(\eta)} f(\zeta)|^2}{1-|f(\zeta)|^2} \leq \frac{1-|f(\eta)|^2}{1-|\eta|^2} \cdot \frac{|1-\bar{\eta}\zeta|^2}{1-|\zeta|^2} \tag{7}$$

for all η, $\zeta \in \Delta$. Now choose a sequence $\{\eta_k\} \subset \Delta$ converging to σ and such that

$$\lim_{k\to+\infty} \frac{1-|f(\eta_k)|}{1-|\eta_k|} = \beta;$$

in particular, $|f(\eta_k)| \to 1$, and so up to a subsequence we can assume that $f(\eta_k) \to \tau \in \partial\Delta$ as $k \to +\infty$. Then setting $\eta = \eta_k$ in 7 and taking the limit as $k \to +\infty$ we obtain 6.

We are left to prove the uniqueness of τ. To do so, we need a geometrical interpretation of 6.

Definition 2.6 *The* horocycle $E(\sigma, R)$ *of center* σ *and radius* R *is the set*

$$E(\sigma, R) = \left\{ \zeta \in \Delta \;\middle|\; \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} < R \right\}.$$

Geometrically, $E(\sigma, R)$ is an euclidean disk of euclidean radius $R/(1+R)$ internally tangent to $\partial\Delta$ in σ; in particular,

$$|\sigma - \zeta| \le \frac{2R}{1+R} < 2R \tag{8}$$

for all $\zeta \in E(\bar{\sigma}, R)$. A horocycle can also be seen as the limit of Poincaré disks with fixed euclidean radius and centers converging to σ (see, e.g., [Ju2] or [A3, Proposition 1.2.1]).

The formula 6 then says that

$$f\big(E(\sigma, R)\big) \subseteq E(\tau, \beta R)$$

for any $R > 0$. Assume, by contradiction, that 6 also holds for some $\tau_1 \neq \tau$, and choose $R > 0$ so small that $E(\tau, \beta R) \cap E(\tau_1, \beta R) =$. Then we get

$$\neq f\big(E(\sigma, R)\big) \subseteq E(\tau, \beta R) \cap E(\tau_1, \beta R) =,$$

contradiction. Therefore 6 can hold for at most one $\tau \in \partial\Delta$, and we are done. □

In Section 4 we shall need a sort of converse of Julia's lemma:

Lemma 2.7 *Let* $f \in \mathrm{Hol}(\Delta, \Delta)$, σ, $\tau \in \partial\Delta$ *and* $\beta > 0$ *be such that*

$$f\big(E(\sigma, R)\big) \subseteq E(\tau, \beta R)$$

for all $R > 0$. *Then* $\beta_f(\sigma) \le \beta$.

Proof. For $t \in [0, 1)$ set $R_t = (1-t)/(1+t)$, so that $t\sigma \in \partial E(\sigma, R_t)$. Therefore $f(t\sigma) \in E(\tau, \bar{\beta} R_t)$; hence

$$\frac{1-|f(t\sigma)|}{1-t} \le \frac{|\tau - f(t\sigma)|}{1-t} < 2\beta \frac{R_t}{1-t} = \frac{2}{1+t}\beta,$$

by 8, and thus

$$\beta_f(\sigma) = \liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} \le \liminf_{t\to1^-} \frac{1-|f(t\sigma)|}{1-t} \le \beta.$$

□

To complete the proof of Theorem 1.1 we still need to give a precise definition of what we mean by non-tangential limit.

Definition 2.8 *Take $\sigma \in \partial\Delta$ and $M \geq 1$; the* Stolz region $K(\sigma, M)$ *of vertex σ and amplitude M is given by*

$$K(\sigma, M) = \left\{ \zeta \in \Delta \,\middle|\, \frac{|\sigma - \zeta|}{1 - |\zeta|} < M \right\}.$$

Geometrically, $K(\sigma, M)$ is an egg-shaped region, ending in an angle touching the boundary of Δ at σ. The amplitude of this angle tends to 0 as $M \to 1^+$, and tends to π as $M \to +\infty$. Therefore we can use Stolz regions to define the notion of non-tangential limit:

Definition 2.9 *A function $f\colon \Delta \to \mathbb{C}$ admits* non-tangential limit $L \in \mathbb{C}$ *at the point $\sigma \in \partial\Delta$ if $f(\zeta) \to L$ as ζ tends to σ inside $K(\sigma, M)$ for any $M > 1$.*

From the definitions it is apparent that horocycles and Stolz regions are strongly related. For instance, if ζ belongs to $K(\sigma, M)$ we have

$$\frac{|\sigma - \zeta|^2}{1 - |\zeta|^2} = \frac{|\sigma - \zeta|}{1 - |\zeta|} \cdot \frac{|\sigma - \zeta|}{1 + |\zeta|} < M|\sigma - \zeta|,$$

and thus $\zeta \in E(\sigma, M|\sigma - \zeta|)$.

We are then ready for the

Proof of Theorem 1.1: Assume that f is β-Julia at σ, fix $M > 1$ and choose any sequence $\{\zeta_k\} \subset K(\sigma, M)$ converging to σ. In particular, $\zeta_k \in E(\sigma, M|\sigma - \zeta_k|)$ for all $k \in \mathbb{N}$. Then Theorem 2.5 gives a unique $\tau \in \partial\Delta$ such that $f(\zeta_k) \in E(\tau, \beta M|\sigma - \zeta_k|)$. Therefore every limit point of the sequence $\{f(\zeta_k)\}$ must be contained in the intersection

$$\bigcap_{k \in \mathbb{N}} E(\tau, \beta \bar{M}|\sigma - \zeta_k|) = \{\tau\},$$

that is $f(\zeta_k) \to \tau$, and we have proved that f has non-tangential limit τ at σ. □

To prove Theorem 1.2 we need another ingredient, known as *Lindelöf principle.* The idea is that the existence of the limit along a given curve in Δ ending at $\sigma \in \partial\Delta$ forces the existence of the non-tangential limit at σ. To be more precise:

Definition 2.10 *Let $\sigma \in \partial\Delta$. A* σ-curve *in Δ is a continous curve $\gamma\colon [0, 1) \to \Delta$ such that $\gamma(t) \to \sigma$ as $t \to 1^-$. Furthermore, we shall say that a function $f\colon \Delta \to \mathbb{C}$ is K-bounded at σ if for every $M > 1$ there exists $C_M > 0$ such that $|f(\zeta)| \leq C_M$ for all $\zeta \in K(\sigma, M)$.*

Then Lindelöf [Li2] proved the following

Theorem 2.11 *Let $f\colon\Delta\to\mathbb{C}$ be a holomorphic function, and $\sigma\in\partial\Delta$. Assume there is a σ-curve $\gamma\colon[0,1)\to\Delta$ such that $f\bigl(\gamma(t)\bigr)\to L\in\mathbb{C}$ as $t\to 1^-$. Assume moreover that*

(a) f is bounded, or that
(b) f is K-bounded and γ is non-tangential, that is its image is contained in a K-region $K(\sigma,M_0)$.

Then f has non-tangential limit L at σ.

Proof. A proof of case (a) can be found in [A3, Theorem 1.3.23] or in [Ru, Theorem 8.4.1]. Since each $K(\sigma,M)$ is biholomorphic to Δ and the biholomorphism extends continuously up to the boundary, case (b) is a consequence of (a). Furthermore, it should be remarked that in case (b) the existence of the limit along γ automatically implies that f is K-bounded ([Li1]; see [Bu, 5.16] and references therein).

However, we shall describe here an easy proof of case (b) when γ is radial, that is $\gamma(t)=t\sigma$, which is the case we shall mostly use.

First of all, without loss of generality we can assume that $\sigma=1$, and then the Cayley transform allows us to transfer the stage to $H^+=\{w\in\mathbb{C}\mid \operatorname{Im} w>0\}$. The boundary point we are interested in becomes ∞, and the curve γ is now given by $\gamma(t)=i(1+t)/(1-t)$.

Furthermore if we denote by $K(\infty,M)\subset H^+$ the image under the Cayley transform of $K(1,M)\subset\Delta$, and by K_ε the truncated cone

$$K_\varepsilon=\bigl\{w\in H^+\mid \operatorname{Im} w>\varepsilon\max\{1,|\operatorname{Re} w|\}\bigr\},$$

we have

$$K(\infty,M)\subset K_{1/(2M)}\qquad\text{and}$$
$$K_{1/(2M)}\cap\{w\in H^+\mid \operatorname{Im} w>R\}\subset K(\infty,M'),$$

for every R, $M>1$, where

$$M'=\sqrt{1+4M^2\frac{R+1}{R-1}}.$$

The first inclusion is easy; the second one follows from the formula

$$\left|\frac{1-\zeta}{1-|\zeta|}\right|^2=1+\frac{2}{|\zeta|+\operatorname{Re}\zeta}\left|\frac{\operatorname{Im}\zeta}{1-|\zeta|}\right|^2, \tag{9}$$

true for all $\zeta\in\Delta$ with $\operatorname{Re}\zeta>0$.

Therefore we are reduced to prove that if $f\colon H^+\to\mathbb{C}$ is holomorphic and bounded on any K_ε, and $f\circ\gamma(t)\to L\in\mathbb{C}$ as $t\to 1^-$, then $f(w)$ has limit L as w tends to ∞ inside K_ε.

Choose $\varepsilon'<\varepsilon$ (so that $K_{\varepsilon'}\supset K_\varepsilon$), and define $f_n\colon K_{\varepsilon'}\to\mathbb{C}$ by $f_n(w)=f(nw)$. Then $\{f_n\}$ is a sequence of uniformly bounded holomorphic functions.

Furthermore, $f_n(ir) \to L$ as $n \to +\infty$ for any $r > 1$; by Vitali's theorem, the whole sequence $\{f_n\}$ is then converging uniformly on compact subsets to a holomorphic function $f_\infty: K_{\varepsilon'} \to \mathbb{C}$. But we have $f_\infty(ir) = L$ for all $r > 1$; therefore $f_\infty \equiv L$. In particular, for every $\delta > 0$ we can find $N \geq 1$ such that $n \geq N$ implies

$$|f_n(w) - L| < \delta \qquad \text{for all } w \in \bar{K}_\varepsilon \text{ such that } 1 \leq |w| \leq 2.$$

This implies that for every $\delta > 0$ there is $R > 1$ such that $w \in \bar{K}_\varepsilon$ and $|w| > R$ implies $|f(w) - L| < \delta$, that is the assertion. Indeed, it suffices to take $R = N$; if $|w| > N$ let $n \geq N$ be the integer part of $|w|$, and set $w' = w/n$. Then $w' \in \bar{K}_\varepsilon$ and $1 \leq |w'| \leq 2$, and thus

$$|f(w) - L| = |f_n(w') - L| < \delta,$$

as claimed. □

Example 1. It is very easy to provide examples of K-bounded functions which are not bounded: for instance $f(\zeta) = (1 + \zeta)^{-1}$ is K-bounded at 1 but it is not bounded in Δ. More generally, every rational function with a pole at $\tau \in \partial\Delta$ and no poles inside Δ is not bounded on Δ but it is K-bounded at every $\sigma \in \partial\Delta$ different from τ.

We are now ready to begin the proof of Theorem 1.2. Let then $f \in \mathrm{Hol}(\Delta, \Delta)$ be β-Julia at $\sigma \in \partial\Delta$, and let $\tau \in \partial\Delta$ be the non-tangential limit of f at σ provided by Theorem 1.1. We would like to show that f' has non-tangential limit $\beta\tau\bar{\sigma}$ at σ; but first we study the behavior of the incremental ratio $\big(f(\zeta) - \tau\big)/(\zeta - \sigma)$.

Proposition 2.12 *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be β-Julia at $\sigma \in \partial\Delta$, and let $\tau \in \partial\Delta$ be the non-tangential limit of f at σ. Then the incremental ratio*

$$\frac{f(\zeta) - \tau}{\zeta - \sigma}$$

is K-bounded and has non-tangential limit $\beta\tau\bar{\sigma}$ at σ.

Proof. We shall show that the incremental ratio is K-bounded and that it has radial limit $\beta\tau\bar{\sigma}$ at σ; the assertion will then follow from Theorem 2.11.(b).

Take $\zeta \in K(\sigma, M)$. We have already remarked that we then have $\zeta \in E(\sigma, M|\zeta - \sigma|)$, and thus $f(\zeta) \in E(\tau, \beta M|\zeta - \sigma|)$, by Julia's Lemma. Recalling 8 we get

$$|f(\zeta) - \tau| < 2\beta M|\zeta - \sigma|,$$

and so the incremental ratio is bounded by $2\beta M$ in $K(\sigma, M)$.

Now given $t \in [0, 1)$ set $R_t = (1 - t)/(1 + t)$, so that $t\sigma \in \partial E(\sigma, R_t)$. Then $f(t\sigma) \in \overline{E(\tau, \beta R_t)}$, and thus

$$1 - |f(t\sigma)| \leq |\tau - f(t\sigma)| \leq 2\beta R_t = 2\beta\frac{1 - t}{1 + t}.$$

Therefore

$$\frac{1-|f(t\sigma)|}{1-t} \leq \left|\frac{\tau - f(t\sigma)}{1-t}\right| \leq \frac{2}{1+t}\beta = \frac{2}{1+t}\liminf_{\zeta\to\sigma}\frac{1-|f(\zeta)|}{1-|\zeta|};$$

letting $t \to 1^-$ we see that

$$\lim_{t\to 1^-}\frac{1-|f(t\sigma)|}{1-t} = \lim_{t\to 1^-}\left|\frac{\tau - f(t\sigma)}{1-t}\right| = \beta, \tag{10}$$

and then

$$\lim_{t\to 1^-}\frac{|\tau - f(t\sigma)|}{1-|f(t\sigma)|} = 1. \tag{11}$$

Since $f(t\sigma) \to \tau$, we know that $\mathrm{Re}\big(\bar\tau f(t\sigma)\big) > 0$ for t close enough to 1; then 9 and 11 imply

$$\lim_{t\to 1^-}\frac{\tau - f(t\sigma)}{1-|f(t\sigma)|} = \tau,$$

and together with 10 we get

$$\lim_{t\to 1^-}\frac{f(t\sigma)-\tau}{t\sigma - \sigma} = \beta\tau\bar\sigma,$$

as desired. □

By the way, the non-tangential limit of the incremental ratio is usually called the *angular derivative* of f at σ, because it represents the limit of the derivative of f inside an angular region with vertex at σ.

We can now complete the

Proof of Theorem 1.2: Again, the idea is to prove that f' is K-bounded and then show that $f'(t\sigma)$ tends to $\beta\tau\bar\sigma$ as $t \to 1^-$.

Take $\zeta \in K(\sigma, M)$, and choose $\delta_\zeta > 0$ so that $\zeta + \delta_\zeta \varDelta \subset \varDelta$. Therefore we can write

$$f'(\zeta) = \tfrac{1}{2\pi i}\int_{|\eta|=\delta_\zeta}\tfrac{f(\zeta+\eta)}{\eta^2}\,d\eta = \tfrac{1}{2\pi i}\int_{|\eta|=\delta_\zeta}\tfrac{f(\zeta+\eta)-\tau}{\zeta+\eta-\sigma}\cdot\tfrac{\zeta+\eta-\sigma}{\eta^2}\,d\eta \tag{12}$$

$$= \tfrac{1}{2\pi}\int_0^{2\pi}\tfrac{f(\zeta+\delta_\zeta e^{i\theta})-\tau}{\zeta+\delta_\zeta e^{i\theta}-\sigma}\left[1-\tfrac{\sigma-\zeta}{\delta_\zeta e^{i\theta}}\right]d\theta. \tag{13}$$

(14)

Now, if $M_1 > M$ and

$$\delta_\zeta = \frac{1}{M}\frac{M_1 - M}{M_1 + 1}|\sigma - \zeta|,$$

then it is easy to check that $\zeta + \delta_\zeta\varDelta \subset K(\sigma, M_1)$; therefore 12 and the bound on the incremental ratio yield

$$|f'(\zeta)| \leq 2\beta M_1\left[1 + M\frac{M_1+1}{M_1-M}\right],$$

and so f' is K-bounded.

If $\zeta = t\sigma$, we can take $\delta_{t\sigma} = (1-t)(M-1)/(M+1)$ for any $M > 1$, and 12 becomes

$$f'(t\sigma) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(t\sigma + \delta_{t\sigma}e^{i\theta}) - \tau}{t\sigma + \delta_{t\sigma}e^{i\theta} - \sigma}\left[1 - \sigma\frac{M-1}{M+1}e^{-i\theta}\right] d\theta.$$

Since $t\sigma + \delta_{t\sigma}\Delta \subset K(\sigma, M)$, Proposition 2.12 yields

$$\lim_{t\to 1^-} \frac{f(t\sigma + \delta_{t\sigma}e^{i\theta}) - \tau}{t\sigma + \delta_{t\sigma}e^{i\theta} - \sigma} = \beta\tau\bar{\sigma}$$

for any $\theta \in [0, 2\pi]$; therefore we get $f'(t\sigma) \to \beta\tau\bar{\sigma}$ as well, by the dominated convergence theorem, and we are done. □

It is easy to find examples of function $f \in \mathrm{Hol}(\Delta, \Delta)$ with $\beta_f(1) = +\infty$.

Example 2. Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be given by $f(z) = \lambda z^k/k$ where $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$ are such that $k > |\lambda|$. Then $\beta_f(1) = +\infty$ for the simple reason that $|f(1)| = |\lambda|/k < 1$; on the other hand, $f'(1) = \lambda$.

Therefore if $\beta_f(\sigma) = +\infty$ both f and f' might still have finite non-tangential limit at σ, but we have no control on them. However, if we assume that $f(\zeta)$ is actually going to the boundary of Δ as $\zeta \to \sigma$ then the link between the angular derivative and the boundary dilation coefficient is much tighter. Indeed, the final result of this section is

Theorem 2.13 *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and $\sigma \in \partial\Delta$ be such that*

$$\limsup_{t\to 1^-} |f(t\sigma)| = 1. \tag{15}$$

Then

$$\beta_f(\sigma) = \limsup_{t\to 1^-} |f'(t\sigma)|. \tag{16}$$

In particular, f' has finite non-tangential limit at σ iff $\beta_f(\sigma) < +\infty$, and then f has non-tangential limit at σ too.

Proof. If the lim sup in 16 is infinite, then $f'(t\sigma)$ cannot converge as $t \to 1^-$, and thus $\beta_f(\sigma) = +\infty$ by Theorem 1.2.

So assume that the lim sup in 16 is finite; in particular, there is $M > 0$ such that $|f'(t\sigma)| \le M$ for all $t \in [0, 1)$. We claim that $\beta_f(\sigma)$ is finite too — and then the assertion will follow from Theorem 1.2 again.

For all t_1, $t_2 \in [0, 1)$ we have

$$|f(t_2\sigma) - f(t_1\sigma)| = \left|\int_{t_1}^{t_2} f'(t\sigma)\, dt\right| \le M|t_2 - t_1|. \tag{17}$$

Now, 15 implies that there is a sequence $\{t_k\} \subset [0, 1)$ converging to 1 and $\tau \in \partial\Delta$ such that $f(t_k) \to \tau$ as $k \to +\infty$. Therefore 17 yields

$$|\tau - f(t\sigma)| \le M(1-t)$$

for all $t \in [0,1)$. Hence

$$\beta_f(\sigma) = \liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} \le \liminf_{t\to 1^-} \frac{1-|f(t\sigma)|}{1-t} \le \liminf_{t\to 1^-} \frac{|\tau - f(t\sigma)|}{1-t} \le M.$$

□

So Julia's condition $\beta_f(\sigma) < +\infty$ is in some sense optimal.

3 Julia's Lemma

The aim of this section is to describe a generalization of Julia's lemma to several complex variables, and to apply it to get a several variables version of Theorem 1.1.

As we have seen, the one-variable Julia's lemma is a consequence of the Schwarz-Pick lemma or, more precisely, of the contracting properties of the Poincaré metric and distance. So it is only natural to look first for a generalization of the Poincaré metric.

Among several such generalizations, the most useful for us is the Kobayashi metric, introduced by Kobayashi [Kob1] in 1967.

Definition 3.1 *Let X be a complex manifold: the* Kobayashi (pseudo)metric *of X is the function $\kappa_X\colon TX \to \mathbb{R}^+$ defined by*

$$\kappa_X(z;v) = \inf\{|\xi| \mid \exists\varphi \in \mathrm{Hol}(\Delta, X) : \varphi(0) = z, d\varphi_0(\xi) = v\}$$

for all $z \in X$ and $v \in T_zX$. Roughly speaking, $\kappa_X(z;v)$ measures the (inverse of) the radius of the largest (not necessarily immersed) holomorphic disk in X passing through z tangent to v.

The Kobayashi pseudometric is an upper semicontinuous (and often continuous) complex Finsler pseudometric, that is it satisfies

$$\kappa_X(z;\lambda v) = |\lambda|\kappa_X(z;v) \tag{18}$$

for all $z \in X$, $v \in T_zX$ and $\lambda \in \mathbb{C}$. Therefore it can be used to compute the length of curves:

Definition 3.2 *If $\gamma\colon [a,b] \to X$ is a piecewise C^1-curve in a complex manifold X then its* Kobayashi (pseudo)length *is*

$$\ell_X(\gamma) = \int_a^b \kappa_X\bigl(\gamma(t);\dot\gamma(t)\bigr)\,dt.$$