In "Buckling" the woody cells are very weakly lignified. Other distinctions will doubtless reveal themselves on a more intensive study.

Tetraploid plants from two distinct races were examined. Both showed an elongation of the trunk and an increase in the width of the first angle, the first pair of internodes and the stem diameter. Internally, the cells were found to be significantly larger than in the normal type and the woody tissue tended to be less strongly lignified.

Only a preliminary survey of the field has as yet been made. The authors believe, however, that these Datura cultures provide exceptionally promising material for a study of the effect of specific factors and of specific chromosomes, particularly upon structural characters; and they hope through further investigation to be able to contribute materially to a factorial analysis of the chromosomes of this species.

[1] Paper presented before the Botanical Society of America, December 28, 1921.

[2] Albert F. Blakeslee, "Types of Mutations and Their Possible Significance in Evolution," *Amer. Naturalist*, **55**, 1921 (254–267); and "Variations in Datura Due to Changes in Chromosome Number," *Ibid.*, Jan.–July, 1922.

THE RIEMANN GEOMETRY AND ITS GENERALIZATION

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1. One of the simplest ways of generalizing Euclidean Geometry is to start by assuming (1) that the space to be considered is an n-dimensional manifold in the sense of Analysis Situs, and (2) that in this space there exists a system of curves called *paths* which, like the straight lines in a euclidean space, serve as a means of finding one's way about.

These paths are defined as the solutions of a system of differential equations,

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{1.1}$$

in which the Γ^i_{jk}'s are analytic functions of $(x^1, x^2, \ldots, x^n)$ and the indices i, j, k run from 1 to n. The second term is a summation with regard to j and k in accordance with the usual convention in such formulas that any term represents a summation with respect to each letter which appears in it both as a subscript and as a superscript.

Since the second term is a quadratic form in $\frac{dx^i}{ds}$, there is no loss of generality in assuming, as we do, that

$$\Gamma^i_{jk} = \Gamma^i_{kj} \tag{1.2}$$

This definition of the *paths* is immediately suggested by the fact that the differential equations of the straight lines in a euclidean space which are

$$\frac{d^2x^i}{ds^2}=0 \tag{1.3}$$

in cartesian coördinates, take the form (1.1) in general coördinates, the Γ's now being such that there shall exist an analytic transformation of $(x_1, x_2, \ldots x_n)$ converting (1.1) into (1.3).

2. This generalized geometry has been studied by H. Weyl in his book, "Raum, Zeit, Materie," Berlin, 1919, and in Vol. 1 of the "Mathematische Zeitschrift." It has also been considered by A. S. Eddington in *Proc. Roy. Soc. London*, **99A** (1921). Both these authors define it in terms of a generalization of Levi-Civita's concept of infinitesimal parallelism rather than by the more natural idea of a system of paths.

It reduces to the Riemann geometry if we assume that there exists a quadratic form

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \qquad (g_{ij}=g_{ji}) \tag{2.1}$$

with respect to which the paths are geodesics, i.e., if the curves for which the integral

$$\int \sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}\,ds \tag{2.2}$$

is stationary satisfy the differential equations (1.1). The conditions that (2.2) be stationary are

$$g_{\alpha i}\Gamma^{\alpha}_{jk} = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x^i}\right). \tag{2.3}$$

When solved for the derivatives of the g's these conditions become

$$\frac{\partial g_{ij}}{\partial x^k} - g_{\alpha j}\Gamma^{\alpha}_{ik} - g_{i\alpha}\Gamma^{\alpha}_{jk} = 0. \tag{2.4}$$

3. In general there exists no quadratic form (2.1) for which the paths are geodesics. For example, in a two-dimensional manifold the system of paths defined by

$$\begin{aligned} \frac{d^2x^1}{ds^2} + (x^1-x^2)\left(\frac{dx^1}{ds}\right)^2 &=0, \\ \frac{d^2x^2}{ds^2} + (x^1-x^2)\left(\frac{dx^2}{ds}\right)^2 &=0 \end{aligned} \tag{3.1}$$

is one for which the equations (2.4) are inconsistent. Therefore there exists no quadratic form (2.1) of which the paths (3.1) are geodesics.

The problem of determining under what conditions the geometry of paths is Riemannian is one of the inverse problems of the Calculus of Variations. But it does not seem hitherto to have been solved. Before considering it further we must mention a few general theorems, which have already been noted more or less explicitly by Weyl and Eddington.

4. If we put $x^i = \varphi^i(\bar{x}^1, \ldots \bar{x}^n)$, thus introducing a new set of coördinates, the equations (1.1) become

$$\frac{d^2\bar{x}^i}{ds^2} + \bar{\Gamma}^i_{jk}\frac{d\bar{x}^j}{ds}\frac{d\bar{x}^k}{ds} = 0, \tag{4.1}$$

where

$$\frac{\partial^2 x^p}{\partial\bar{x}^i\partial\bar{x}^j} + \Gamma^p_{qr}\frac{\partial x^q}{\partial\bar{x}^i}\frac{\partial x^r}{\partial\bar{x}^j} = \bar{\Gamma}^t_{ij}\frac{\partial x^p}{\partial\bar{x}^t}. \tag{4.2}$$

Expressing the conditions of integrability of these equations regarded as differential equations for determining the x's as functions of the $\bar{x}$'s, when the Γ's and $\bar{\Gamma}$'s are known, we obtain

$$\frac{\partial x^q}{\partial\bar{x}^i}\frac{\partial x^r}{\partial\bar{x}^j}\frac{\partial x^s}{\partial\bar{x}^k}B^p_{qrs} = \frac{\partial x^p}{\partial\bar{x}^t}\bar{B}^t_{ijk} \tag{4.3}$$

where

$$B^p_{qrs} = \frac{\partial\Gamma^p_{qs}}{\partial x^r} - \frac{\partial\Gamma^p_{qr}}{\partial x^s} + \Gamma^p_{\alpha r}\Gamma^\alpha_{qs} - \Gamma^p_{\alpha s}\Gamma^\alpha_{qr}. \tag{4.4}$$

Equation (4.3) may also be written

$$\frac{\partial x^q}{\partial\bar{x}^i}\frac{\partial x^r}{\partial\bar{x}^j}\frac{\partial x^s}{\partial\bar{x}^k}\frac{\partial\bar{x}^t}{\partial x^p}B^p_{qrs} = \bar{B}^t_{ijk}. \tag{4.5}$$

This states that the functions $\bar{B}$ are the components in the coördinate system $\bar{x}$ of a tensor, B, which is contravariant of the first order and covariant of the third order, with respect to the group of all transformations of the paths into themselves. This tensor is called the *curvature tensor* of the manifold.

From (4.4) it follows that

$$B^p_{qrs} + B^p_{qsr} = 0, \tag{4.6}$$

and

$$B^p_{qrs} + B^p_{rsq} + B^p_{sqr} = 0. \tag{4.7}$$

The theory of covariant differentiation (cf. Ricci and Levi-Civita in *Math. Ann.*, **54** (1901) can be generalized at once to the geometry of paths by replacing the Christoffel symbols $\left\{ {jk \atop i} \right\}$ by the functions Γ^i_{jk} in all formulas. In particular it is easily proved by means of (4.1) and (4.2) that the operation of covariant differentiation converts any tensor into a tensor of higher order. The formulas for covariant differentiation of sums and products of tensors also generalize without change, and also the theorem that if a_{ij} is any tensor and a_{ijkl} is its second covariant derivative,

$$a_{ijkl} - a_{ijlk} = a_{\alpha j}B^\alpha_{ikl} + a_{i\alpha}B^\alpha_{jkl}. \tag{4.8}$$

5. Returning now to the question as to the conditions on the Γ's that they shall yield a Riemann geometry, we observe that the left hand member of (2.4) is the covariant derivative of g_{ij}. Thus (2.4) may be written

$$g_{ijk} = 0. \tag{5.1}$$

By (4.8)

$$g_{ijkl} - g_{ijlk} = g_{\alpha j} B^{\alpha}_{ikl} + g_{i\alpha} B^{\alpha}_{jkl} \tag{5.2}$$

which combined with (5.1) gives

$$g_{\alpha j} B^{\alpha}_{ikl} + g_{i\alpha} B^{\alpha}_{jkl} = 0. \tag{5.3}$$

If these equations be differentiated covariantly, we get

$$g_{i\alpha} B^{\alpha}_{jklm} + g_{\alpha j} B^{\alpha}_{iklm} + g_{i\alpha m} B^{\alpha}_{jkl} + g_{\alpha jm} B^{\alpha}_{ikl} = 0. \tag{5.4}$$

Hence by (5.1)

$$g_{i\alpha} B^{\alpha}_{jklm} + g_{\alpha j} B^{\alpha}_{iklm} = 0. \tag{5.5}$$

Proceeding in this manner we get a sequence of equations, namely

$$\begin{aligned} g_{i\alpha} B^{\alpha}_{jklmn} + g_{\alpha j} B^{\alpha}_{iklmn} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ g_{i\alpha} B^{\alpha}_{jklmn \ldots p} + g_{\alpha j} B^{\alpha}_{iklmn \ldots p} &= 0. \end{aligned} \tag{5.6}$$

If the geometry of paths is Riemannian (5.1) must be satisfied and hence also (5.3), (5.5) and (5.6). The equations (5.3), (5.5) and (5.6) are linear equations in the $(n + 1)n/2$ functions g_{ij} with coefficients which are functions of the Γ's alone. Hence the algebraic conditions for the consistency of (5.3), (5.5) and (5.6) regarded as linear equations in the g's are necessary conditions on the Γ's that the geometry of paths shall be Riemannian.

6. Now suppose that equations (5.3) and (5.5) are algebraically consistent in the g's, and that the rank of the matrix of the B's is such that the g's are determined by (5.3) to within a factor, which is at most a function of the x's. Let g_{ij} stand for a particular solution of (5.3) and (5.5). If (5.3) be differential covariantly with respect to x_m, we have in consequence of (5.5)

$$g_{i\alpha m} B^{\alpha}_{jkl} + g_{\alpha jm} B^{\alpha}_{ikl} = 0. \tag{6.1}$$

Since these equations are of the same form as (5.3), it follows from the above hypothesis about (5.3) that

$$g_{ijk} = \varphi_k g_{ij}, \tag{6.2}$$

where φ_k is a covariant vector. Substituting these expressions in

$$g_{ijkl} - g_{ijlk} = 0, \tag{6.3}$$

which follows from (5.2) and (5.3), we obtain

$$\frac{\partial \varphi_l}{\partial x^k} = \frac{\partial \varphi_k}{\partial x^l}.$$

Hence the vector φ_k is the gradient of a scalar function, and can be put in the form

$$\varphi_k = -\frac{\partial \log \lambda}{\partial x^k}. \tag{6.4}$$

Substituting this value of φ_k in the right member of (6.2) and the explicit expression for g_{ijk} (cf. 2.4) in the left member, we obtain

$$\frac{\partial}{\partial x^k}(\lambda g_{ij}) - \lambda g_{pi}\Gamma^p_{jk} - \lambda g_{pj}\Gamma^p_{ik} = 0. \tag{6.5}$$

Consequently the functions λg_{ij} satisfy (2.4) and give a Riemann geometry.

It is interesting to note that the geometry of paths obtained from (6.2) without imposing the condition (6.4) is the geometry used by Weyl as the basis for a combined electromagnetic and gravitational theory. For (6.2) is equivalent to

$$g_{\alpha k}\Gamma^\alpha_{ij} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) + \frac{1}{2}(g_{ik}\varphi_j + g_{jk}\varphi_i - g_{ij}\varphi_k). \tag{6.6}$$

7. Let us now assume that (5.3) are algebraically consistent, and that all of their solutions satisfy (5.5). Let $g^{(1)}_{ij}, g^{(2)}_{ij} \ldots\ldots g^{(p)}_{ij}$ be a complete set of solutions. The general solution is expressible in the form

$$g_{ij} = \varphi^{(1)} g^{(1)}_{ij} + \varphi^{(2)} g^{(2)}_{ij} + \ldots + \varphi^{(p)} g^{(p)}_{ij}. \tag{7.1}$$

Differentiating (5.3) covariantly, we get (6.1), and consequently we must have

$$g^{(\alpha)}_{ijk} = \lambda^{(\alpha 1)}_k g^{(1)}_{ij} + \ldots + \lambda^{(\alpha p)}_k g^{(p)}_{ij}. \tag{7.2}$$

The p^2 vectors $\lambda^{(\alpha\beta)}_k$ $(\alpha, \beta = 1, \ldots p)$ must be such that the functions $g^{(\alpha)}_{ijk}$ satisfy (6.3). On substituting (7.2) in (6.3) we find

$$\frac{\partial \lambda^{(\alpha\beta)}_k}{\partial x^l} - \frac{\partial \lambda^{(\alpha\beta)}_l}{\partial x^k} + \sum_{\gamma=1}^{p}\left(\lambda_k{}^{(\alpha\gamma)}\lambda_l{}^{(\gamma\beta)} - \lambda_l{}^{(\alpha\gamma)}\lambda_k{}^{(\gamma\beta)}\right) = 0. \tag{7.3}$$

When we express the condition that the functions g_{ij} given by (7.1) shall satisfy the conditions $g_{ijk} = 0$, we find in consequence of (7.2) that the functions $\varphi^{(\alpha)}$ must satisfy the equations

$$\frac{\partial \varphi^{(\alpha)}}{\partial x^k} + \sum_{\beta=1}^{p} \varphi^{(\beta)}\lambda_k{}^{(\beta\alpha)} = 0. \tag{7.4}$$

In consequence of (7.3) this system of equations is completely integrable, and hence there exists a set of φ's which by means of (7.1) determine a system of g's which yield a Riemann geometry. Hence we have the theorem: *In order that the geometry of paths shall be a Riemann geometry it is sufficient that the* Γ's *be such that the equations (5.3) be algebraically consistent, and that all of their solutions satisfy (5.5).*